Nash Equilibrium

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When players beliefs are correct

Consider the two-player game of Figure 1. There are no dominance relationships for either player and therefore all pure outcomes survive the iterated elimination of strictly dominated strategies. Because there are only two players, this is also the set of rationalizable pure strategy profiles.[1] Consider the strategy profile (U,r). We can alternatively establish that this outcome is rationalizable by performing an explicit analysis of the players' beliefs: U is a best response by Row if she believes that Column is choosing l. Column's choice of l would be rationalized by his belief that Row were playing U. Therefore the consistent set of beliefs which rationalizes U is[2,3]

$\mathscr{R}\ (U)$	$\mathscr{R}$ plays U,	(1a)
$\mathscr{R}\ \mathscr{C}\ (l)$	$\mathscr{R}$ believes $\mathscr{C}$ will play l,	(1b)
$\mathscr{R}\ \mathscr{C}\ \mathscr{R}\ (U)$	$\mathscr{R}$ believes $\mathscr{C}$ believes $\mathscr{R}$ will play U.	(1c)

Similarly, r is rationalizable for Column because it is a best response if he believes that Row will play D, and D is a best response by Row if she believes that Column will choose right. Therefore the consistent set of beliefs which rationalizes r is

$\mathscr{C}\ (r)$	$\mathscr{C}$ plays r,	(2a)
$\mathscr{C}\ \mathscr{R}\ (D)$	$\mathscr{C}$ believes $\mathscr{R}$ will play D,	(2b)
$\mathscr{C}\ \mathscr{R}\ \mathscr{C}\ (r)$	$\mathscr{C}$ believes $\mathscr{R}$ believes $\mathscr{C}$ will play r.	(2c)

1 See the handout "Nonequilibrium Solution Concepts: Iterated Dominance and Rationalizability."

2 See the handout "Nonequilibrium Solution Concepts: Iterated Dominance and Rationalizability."

3 $\mathscr{R}(U)$ is not a belief; but the $\mathscr{R}\mathscr{C}\cdots(\cdot)$ expressions below it are beliefs.

	l	r
U	**5**,**5**	4,4
D	4,4	**5**,**5**

Figure 1.

After this game is played in this way—viz. Row plays Up and Column plays right—each player will realize *ex post* that her beliefs about her opponent's play were incorrect and, further, each will regret her own choice in the light of what she learned about her opponent's strategy. Specifically, from (1b) we see that Row believed that Column would play l, but Column instead chose r. Had Row known that Column would choose r, she would have chosen Down instead. Similarly, from (2b) we see that Column believed that Row would play Down, but Row played Up instead. Had Column known that Row would play Up, he would have preferred to have chosen left. In this (U,r) outcome, then, each player was choosing a best response to her beliefs about the strategy of her opponent, but each player's beliefs were wrong.

Now consider the strategy profile (U,l). We have already seen [from (1a) → (1c) above] that Up is rationalizable for Row. To see that left is rationalizable for Column we need only exhibit the following consistent set of beliefs:

$\mathscr{C}\quad(l)$	$\mathscr{C}$ plays l,	(3a)
$\mathscr{C}\quad\mathscr{R}\quad(U)$	$\mathscr{C}$ believes $\mathscr{R}$ will play U,	(3b)
$\mathscr{C}\quad\mathscr{R}\quad\mathscr{C}\quad(U)$	$\mathscr{C}$ believes $\mathscr{R}$ believes $\mathscr{C}$ will play l.	(3c)

When the game is played this way—viz. Row plays Up and Column plays left—each player's prediction of her opponent's strategy was indeed correct. And since each player was playing a best response to her correct beliefs, neither player regrets her own choice of strategy.

In other words, when rational players correctly forecast the strategies of their opponents they are not merely playing best responses to their *beliefs* about their opponents' play; they are playing best responses to the *actual* play of their opponents. When all players correctly forecast their opponents' strategies, and play best responses to these forecasts, the resulting strategy profile is a *Nash equilibrium*.[4] (See Nash [1951].)

Before defining Nash equilibrium, let's quickly recap our notation. The player set is $I=\{1,\dots,n\}$. Each player i's pure-strategy space is S_i and her mixed-strategy space is Σ_i (the set of probability distributions over S_i). When these symbols lack subscripts, they refer to Cartesian products over the player set. A subscript of "$-i$" indicates the set $I\backslash\{i\}$. Her expected utility from a mixed-strategy profile $\boldsymbol{\sigma}$ is $u_i(\boldsymbol{\sigma})$.

4 Note in Figure 1 that both players payoffs in the (U,l) box are bolded. This indicates that Row's payoff is maximal giving Column's choice and that Column's choice is maximal given Row's choice. A pure-strategy profile is a Nash equilibrium if and only if its payoff vector has every element in boldface. Similarly, (D,r) is also a Nash equilibrium of this game.

Definition A *pure-strategy Nash equilibrium* of a strategic-form game is a pure-strategy profile $\boldsymbol{s}^* \in S$ such that "every player is playing a best response to the strategy choices of her opponents." More formally, we say that $\boldsymbol{s}^*$ is a Nash equilibrium if

$$(\forall i \in I) \;\square\; s_i{}^* \text{ is a best response to } \boldsymbol{s}^*_{-i}, \tag{4a}$$

or, equivalently,

$$(\forall i \in I) \;\square\; s_i{}^* \in \mathsf{BR}_i(\boldsymbol{s}^*_{-i}), \tag{4b}$$

or, more notationally,

$$(\forall i \in I)\ (\forall s_i \in S_i) \;\square\; u_i(s_i{}^*, \boldsymbol{s}^*_{-i}) \geq u_i(s_i, \boldsymbol{s}^*_{-i}). \tag{4c}$$

Note well that when a player i judges the optimality of her part of the equilibrium prescription—i.e. decides whether *she* will play her part of the prescription—she does assume that her *opponents* will play their part $\boldsymbol{s}^*_{-i}$ of the prescription. Therefore in (4c) she is asking herself the question: Does there exist a *unilateral* deviation s_i for me such that I would *strictly* gain from such defection given that the opponents held truly to their prescriptions.

A game need not have a pure-strategy Nash equilibrium. Consider the matching pennies game of Figure 2. Each player decides which side of a coin to show. Row prefers that the coins match; Column prefers that they be opposite. We can see from the figure that this game has no pure-strategy equilibrium.[5] No matter how the players think the game will be played (i.e. what pure-strategy profile will be played), one player will always be distinctly unhappy with her choice and would prefer to change her strategy.

	H	T
H	$\mathbf{1}, -1$	$-1, \mathbf{1}$
T	$-1, \mathbf{1}$	$\mathbf{1}, -1$

Figure 2: Matching pennies does not admit a pure-strategy Nash equilibrium.

This nonexistence problem when we restrict ourselves to pure strategies was historically a major motivation for the introduction of mixed strategies into game theory: We will see that the existence of a (possibly degenerate) mixed-strategy Nash equilibrium *is* guaranteed. Here's the natural generalization to mixed strategies of the previous definition:

Definition

A *Nash equilibrium* of a strategic-form game is a mixed-strategy profile $\boldsymbol{\sigma}^* \in \Sigma$ such that "every player is playing a best response to the strategy choices of her opponents." More formally, we say that $\boldsymbol{\sigma}^*$ is a Nash equilibrium if

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5 In no cell of the matrix are both payoffs bolded.

$$(\forall i \in I)\ \square\ \sigma_i^* \text{ is a best response to } \boldsymbol{\sigma}_{-i}^*, \tag{5a}$$

or, equivalently,

$$(\forall i \in I)\square\ \mathsf{supp}\,\sigma_i^* \subset \mathsf{BR}_i(\boldsymbol{\sigma}_{-i}^*), \tag{5b}$$

or, more notationally,[6]

$$(\forall i \in I)\ (\forall s_i \in S_i)\ \square\ u_i(\sigma_i^*, \boldsymbol{\sigma}_{-i}^*) \geq u_i(s_i, \boldsymbol{\sigma}_{-i}^*). \tag{5c}$$

Game Theory for \$60: If Nash equilibrium is the answer, what is the question?

Nash equilibria as self-enforcing agreements

Even though rationalizability would seem to be logically prior to the Nash equilibrium concept (owing to fewer assumptions), Nash equilibrium chronologically predates rationalizability in the development of game theory. Nash equilibrium has been and still is widely employed in applications. So strong is the bias toward Nash equilibrium as virtually a definition of rationality that a common theme in the literature holds that a player who does not play her part of the relevant equilibrium must be making a mistake. Only relatively recently have game theorists regularly expressed serious concern over its justification. (See Kreps [1989] and Kreps [1990].)

The most serious attempted justification of Nash equilibrium is its interpretation as a necessary condition for a *self-enforcing agreement.* Consider a scenario where you and I have an opportunity prior to playing the game to communicate and to reach a nonbinding agreement about how we will play. If we do indeed reach such an agreement, there is a strong argument that the agreement we reach should constitute a Nash equilibrium: Because the agreement is nonbinding, we each have the opportunity, regardless of what we agreed to, to take the selfishly best action given our expectations of what the other will do. If the agreement were not a Nash equilibrium, at least one of us would have an incentive to deviate from the agreement (assuming that that person still believed that the other would carry through with her part of the agreement's specification).

6 To see why (5b) is a translation of (5a), recall that a mixed strategy σ_i is a best response to a deleted mixed-strategy profile $\boldsymbol{\sigma}_{-i}$ if and only if it puts positive weight only upon pure strategies which are themselves best responses to $\boldsymbol{\sigma}_{-i}$. It might not be obvious why (5c) is a sufficient characterization of what we mean when we say that σ_i^* is a best response to $\boldsymbol{\sigma}_{-i}$. It requires that σ_i^* is at least as good as any other pure strategy which i has; however, it doesn't address the possibility that player i might have some even better *mixed* strategy. The key is that player i's payoff to any mixed strategy is a convex combination of her payoffs to her pure strategies. If her payoff to the mixed strategy σ_i^* is weakly greater than each of her pure strategy payoffs, it weakly exceeds any convex combination of these pure-strategy payoffs.

	l	r
U	**9,9**	0,8
D	8,0	**7,7**

Figure 3: A Nash equilibrium need not be self-enforcing.

Robert Aumann [1990] has offered an example to show that a Nash equilibrium need not be self-enforcing. I.e. although being a Nash equilibrium may be necessary for an outcome to be self-enforcing, it is not sufficient. Consider the game of Figure 3. There are two pure-strategy Nash equilibria: (U, l) and (D, r). Both pure-strategy equilibrium profiles have merit as a prediction of play. (U, l) is Pareto dominant, but (D, r) is safer because each player would guarantee herself at least 7 by conforming with this equilibrium profile, while conforming with (U, l) risks a zero payoff if her opponent does not conform. I.e. unless Row is quite certain that her opponent will choose his part, viz. l, of the Pareto-dominant equilibrium, Down yields Row a higher expected payoff than Up. Specifically, playing Up requires that Row attaches a probability of at least &/8 to the event that Column chooses left. A symmetric argument shows that Column must be very certain, in order that he play left, that Row will choose Up.

In fact, Row realizes that Column is reasoning in the same way that she is: that Column will play r unless he is highly certain that Row will play U. This increases Row's scepticism that Column will play l and therefore makes D even more tempting for Row. And of course Column realizes this, so r is even more tempting for him. And Column realizes this, etc. Aumann's point is not that (D, r) must be played by rational players, but rather that it can be plausibly played.

Can preplay communication help the players transcend their lack of confidence and achieve the Pareto-dominant equilibrium? Aumann says no. Consider the case where each player is skeptical enough of the other's intention that in the absence of a preplay agreement they would each choose their part of the (D, r) equilibrium. Regardless of what Column chooses to play, he always prefers that Row choose Up (because $9 > 0$ and $8 > 7$); and Row always prefers that Column choose left. Even if the two players agree prior to the game to play the (U, l) equilibrium, Row might reason this way: If we hadn't had our chat, I would have chosen Down. Now that we have had our chat, do I have reason to be more confident that Column will play left so that I can play Up? No. Regardless of Column's true intentions he would want me to *believe* that he would choose left so that I would play Up. Therefore I have learned nothing from his professed agreement to play left. Column would reason similarly about Row's professed agreement to playing the (U, l) equilibrium. Neither player learns anything from the preplay communication and they each know that the other has learned nothing, etc. Therefore they each remain as skeptical as they were initially and therefore the Pareto-dominated equilibrium (D, r) is played.

This argument does not apply to all games. Consider the game of Figure 4. There are two Nash equilibria which are Pareto unranked; i.e. Row prefers one to the other and Column prefers the other to the one. Each Nash equilibrium Pareto dominates every nonequilibrium outcome. The players would prefer to coordinate on any Nash equilibrium—even the player for whom that equilibrium is not the better—than fail to coordinate on any equilibrium. (This is why this is called a coordination game. If one player believes that one equilibrium is being played and the other player believes the other equilibrium is being played, then no equilibrium will be played.[7])

	l	r
U	**2,1**	0,0
D	0,0	**1,2**

Figure 4: A coordination game, where Nash equilibria are self-enforcing.

Assume that the two players agree prior to the game to play the best-for-Row equilibrium (U, l). Should Row believe Column's claim that he will play left? In this game it is not the case that Column wants Row to play Up regardless of what Column himself plans to do. If Column were to play right, he would prefer that Row play Down rather than Up. By agreeing to (U, l) Column is signaling that he wants Row to play Up. Unlike the game of Figure 3, however, he is also signaling that he plans to keep his part of the agreement as well. Therefore both of the pure-strategy Nash-equilibrium profiles are self-enforcing in this game.

Nash equilibrium as the result of a dynamic process

Nash equilibrium is also widely applied in games where there is no explicit communication and negotiation phase. Justifications for the application of the Nash concept in these contexts is less well developed but is a subject of current research. Recall our study of the rationalizable outcome (U, r), which was not a Nash equilibrium in the game of Figure 1. The occurrence of such an outcome can only be explained if at least one of the players harbors misconceptions about how the game will be played. Furthermore, at least one of the players will express regret about her strategy choice after the end of the game. This suggests that Nash equilibrium would be the relevant solution concept if there were some pregame dynamic which assured that players' beliefs were in agreement prior to their strategy selection. It is problematic to construct such examples, however. As Bernheim [1984] argues[8]

> Specifically, it is fruitless to argue that repetitions of a game generate convergence to equilibrium. Unlike the stylized dynamics of competitive equilibrium, where the movement of prices forms a structural link between repetitions of an economy, there is nothing structural tying together successive plays of a game. Thus, if players are unaware that the game is being repeated, there is no meaningful dynamic. On the other hand, if they are aware of the repetitions, then the repeated game is itself a new game, entirely distinct from its components. Convergence of component choices may then have nothing whatsoever to do with attainment of equilibrium in the game actually played. Attempts to introduce equilibrating forces simply generate larger composite games, and the nature of strategic choices in these larger games remains inherently one-shot.

7 Note that this occurred in the play generated by the system of beliefs (1) and (2) in the game of Figure 1: Row believed the (U, l) was being played and Column believed that (D, r) was being played.

8 We will study in detail games played in a dynamic context later in the semester.

There are three reasons I can think of why you should be very serious about learning about Nash equilibrium: 1 Even though the opportunity for pregame communication and negotiation is not universally available, the class of games in which it is a possibility is an important one; 2 Current attempts to satisfactorily justify the application of Nash equilibrium to a wider class of games may ultimately prove successful (in which case the equilibria of these games will be relevant); and 3 Nash equilibrium is widely applied in economics; any serious economist needs to understand the concept and related techniques very well.

Nash equilibria can be vulnerable to multiplayer deviations

As noted above, the definition of Nash equilibrium only requires the absence of any profitable unilateral deviations by any player. A Nash equilibrium is not guaranteed to be invulnerable to deviations by coalitions of players however. Consider the three-player game of Figure 5. Player 1 chooses a row, player 2 chooses a column, and player 3 chooses a matrix. There are two pure-strategy Pareto-ranked Nash equilibria: (U, l, A) and (D, r, B), where (U, l, A) Pareto dominates (D, r, B).

	l	r
U	**0, 0, 10**	−5, −5, 10
D	−5, −5, 0	**1, 1,** −5

Matrix A

	l	r
U	**−2, −2,** 0	−5, −5, 0
D	−5, −5, **0**	−**1,** −**1, 5**

Matrix B

Figure 5: A three-player game (Row, Column, Matrix)

Consider the (U, l, A) equilibrium. No player wants to deviate unilaterally: Given that Column is choosing l and Matrix is choosing A, Row would be worse off to switch to D. Given that Row and Matrix are choosing U and A, respectively, Column would be worse off choosing r. And given that Row and Column are choosing U and l, respectively, Matrix would be worst off choosing B.

However, fix Matrix's choice at A and consider the joint deviation by Row and Column from (U, l) to (D, r). Both would profit from such a shift in their strategies, yet (U, l, A) is still a Nash equilibrium, because Nash equilibrium is only concerned with the existence of profitable unilateral deviations. A strategy profile is a *strong equilibrium* if no coalition (including the grand coalition, i.e. all the players collectively) can profitably deviate from the prescribed profile. (See Aumann [1959], Aumann [1960].) The definition immediately implies that any strong equilibrium is both Pareto efficient and a Nash equilibrium. A strong equilibrium need not exist.[9] However, note that (D, r, A) to which the coalition of Row and Column might defect is itself not even a Nash equilibrium. Therefore one could question whether it should be used as the basis for rejecting (U, l, A). See Bernheim, et al. [1987] for more on *coalition-proof* Nash equilibrium.

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9 In this game it is clear that a pure-strategy strong equilibrium does not exist. We already showed how (U, l, a) is ruled out. The only other pure-strategy Nash equilibrium, viz. (D, r, B), is ruled out because it is not Pareto efficient.

Existence of Nash equilibrium

Nash equilibrium $\Leftrightarrow$ fixed point of the best-response correspondence

We will now prove that every game has a Nash equilibrium when we allow mixed strategies.[10] We first show that a strategy profile is a Nash equilibrium if and only if it is a *fixed point* of a best-response correspondence. Then we show that this correspondence must have a fixed point.

Let's briefly recall what a fixed point of a function is. Consider a function $f: X \to X$ whose domain is identical with its target set. We say that the element $x \in X$ of the domain is a *fixed point* of the function f if $f(x)=x$. In other words, the function f leaves the point x untransformed.

The concept of a fixed point can be generalized to correspondences.[11] Let the correspondence $\varphi: X \twoheadrightarrow X$ have a domain identical to its target set. We cannot usefully stipulate that $x \in X$ is fixed point of φ if $\varphi(x)=x$ because $\varphi(x)$ is a subset of X and x is an element of X and therefore this equality cannot possibly hold. Instead we say that $x \in X$ is a fixed point of the correspondence $\varphi: X \twoheadrightarrow X$ if $x \in \varphi(x)$. In other words the correspondence's values at a fixed point include the fixed point itself. See Figure 6.

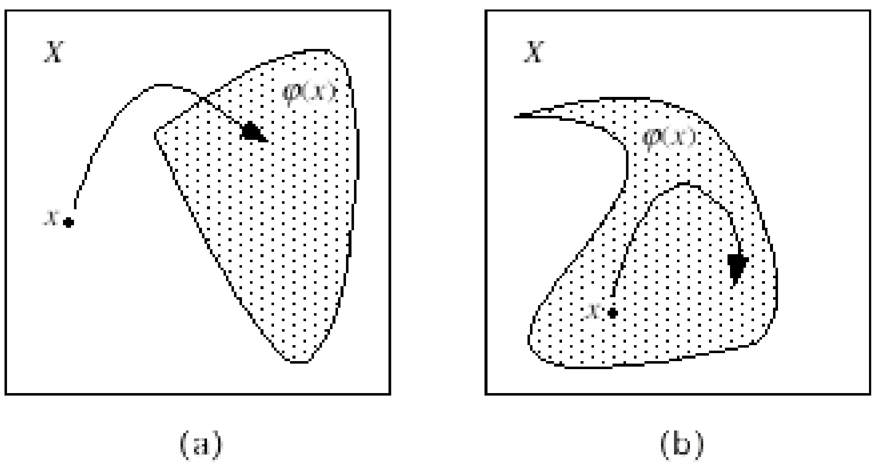

Figure 6: Representations of a correspondence $\varphi: X \twoheadrightarrow X$ when (a) x is not a fixed point of φ and (b) x is a fixed point of φ.[12]

We have previously defined for each player $i \in I$ a pure-strategy best-response correspondence $\mathsf{BR}_i: \Sigma_{-i} \twoheadrightarrow S_i$, which specifies for every deleted mixed-strategy profile $\boldsymbol{\sigma}_{-i} \in \Sigma_{-i}$ by player i's opponents a set $\mathsf{BR}_i(\boldsymbol{\sigma}_{-i}) \subset S_i$ of player-i pure strategies which are best responses. Then we saw that any player-i mixed strategy which put positive weight upon only these pure-strategy best responses was itself a best-response mixed strategy for player i. To prove the existence of a Nash equilibrium we will find it useful

10 Fudenberg and Tirole [1991] is a good reference for the proof of the existence of a Nash equilibrium.

11 Recall that a *correspondence* $\varphi: X \twoheadrightarrow Y$ is a "set-valued function" or, more properly, a mapping which associates to every element $x \in X$ in the domain a *subset* of the target set Y. In other words, $\forall x \in X$, $\varphi(x) \subset Y$.

12 The arrow is not meant to indicate that the point $x \in X$ gets mapped to a single point in the shaded region. Rather x is mapped by φ into the entire shaded region. If the arrows are obfuscating, just ignore them.

to work directly with the mixed-strategy best-response correspondences implied by the pure-strategy best-response correspondences BR_i, $i\in I$.

For each player $i\in I$, we define her mixed-strategy best-response correspondence ψ_i: $\Sigma\to\Sigma_i$ by

$$\psi_i(\boldsymbol{\sigma})=\{\sigma_i{}'\in\Sigma_i\colon \operatorname{supp}\sigma_i{}'\subset \mathsf{BR}_i(\boldsymbol{\sigma}_{-i})\}. \tag{6}$$

In other words given any mixed-strategy profile $\boldsymbol{\sigma}\in\Sigma$ we can extract the deleted mixed-strategy profile $\boldsymbol{\sigma}_{-i}\in\Sigma_{-i}$ from it.[13] Then we determine player i's pure-strategy best responses to $\boldsymbol{\sigma}_{-i}$ and form the set of player-i mixed strategies which put positive weight only upon these pure-strategy best responses. This set of player-i mixed strategies is the value of the player-i mixed-strategy best-response correspondence ψ_i evaluated at $\boldsymbol{\sigma}$.

Now we form a new correspondence ψ by forming the Cartesian product of the n personal mixed-strategy best-response correspondences ψ_i. We define for every $\boldsymbol{\sigma}\in\Sigma$,

$$\psi(\boldsymbol{\sigma})=\mathop{\times}_{i\in I}\psi_i(\boldsymbol{\sigma}). \tag{7}$$

For each $i\in I$, $\psi_i(\boldsymbol{\sigma})\subset\Sigma_i$, so for each $\boldsymbol{\sigma}\in\Sigma$, $\psi(\boldsymbol{\sigma})$ is a subset of the Cartesian product of the individual-player mixed-strategy spaces Σ_i; i.e. $\psi(\boldsymbol{\sigma})\subset\Sigma$. Therefore we see that ψ is a correspondence itself from the space of mixed-strategy profiles into the space of mixed-strategy profiles; i.e. ψ: $\Sigma\twoheadrightarrow\Sigma$.[14]

Consider any Nash equilibrium $\boldsymbol{\sigma}\in\Sigma$. Each player i's mixed strategy σ_i is a best response to the other players' deleted mixed-strategy profile $\boldsymbol{\sigma}_{-i}$. Therefore σ_i satisfies the requirements for inclusion in the set $\psi_i(\boldsymbol{\sigma})$ as defined in (6); i.e. σ_i belongs to player i's mixed-strategy best-response correspondence evaluated at the Nash-equilibrium profile $\boldsymbol{\sigma}$. Because this inclusion must hold for all players, we have

$$\boldsymbol{\sigma}=(\sigma_1,\ldots,\sigma_n)\in\mathop{\times}_{i\in I}\psi_i(\boldsymbol{\sigma})=\psi(\boldsymbol{\sigma}), \tag{8}$$

or in other words if $\boldsymbol{\sigma}$ is a Nash equilibrium we must have

$$\boldsymbol{\sigma}\in\psi(\boldsymbol{\sigma}). \tag{9}$$

I.e. a Nash equilibrium profile is a fixed point of the best-response correspondence ψ. This logic is reversible: any fixed point of the best-response correspondence is a Nash equilibrium profile. Therefore a mixed-strategy profile is a Nash equilibrium if and only if it is a fixed point of the best-response correspondence ψ.

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13 We *could* define the domain for the correspondence ψ_i to be Σ_{-i} rather than Σ. However, we are free to define it the way we do; the definition ignores the extraneous information $\sigma_i\in\Sigma_i$. You will see the formal advantage to this definition as we proceed.

14 Creating a correspondence which maps one set into itself was the motivation behind defining the domains of the personal best-response correspondences ψ_i to be Σ rather than Σ_{-i}.

The best-response correspondence has a fixed point

To prove the existence in general of a Nash equilibrium, we will prove the existence of a fixed point of the best-response correspondence using Kakutani's Kakutani [1941] fixed-point theorem:[15]

Theorem Let $K \subset \mathbb{R}^m$ be compact and convex.[16] Let the correspondence $\varphi\colon K \twoheadrightarrow K$ be upper hemicontinuous with nonempty convex values.[17] Then φ has a fixed point.

In our application of Kakutani's theorem the space Σ of mixed-strategy profiles will play the role of K and the best-response correspondence $\psi\colon \Sigma \twoheadrightarrow \Sigma$ will play the role of φ. We need to verify that Σ and ψ do indeed satisfy the hypotheses of Kakutani's theorem. Then we can claim that the best-response correspondence has a fixed point and therefore every game has a Nash equilibrium.

We need to show that Σ is compact and convex. The space Σ of mixed-strategy profiles is the Cartesian product of the players' mixed-strategy spaces Σ_i, each of which is compact and convex. This will imply that Σ is itself compact and convex. We will prove here that the convexity of the Σ_i implies the convexity of Σ. You can prove as an exercise that the compactness of the Σ_i implies that Σ is compact.

Lemma Let $A_1, A_2, \ldots, A_m$ be convex sets. Let A be their Cartesian product $A \equiv \mathsf{X}_{k=1}^{m} A_k$. Then A is a convex set.

Proof To show that A is convex, we take an arbitrary pair of its members, viz. $a', a'' \in A$, and show that an arbitrary convex combination of this pair, viz. $\hat{a} \equiv \alpha a' + (1-\alpha)a''$, for $\alpha \in [0,1]$, also belongs to A. Both a' and a'' are m-tuples of elements which belong to the constituent sets A_i; i.e. $a' = (a_1', \ldots, a_m')$ and $a'' = (a_1'', \ldots, a_m'')$ where, for each $i \in \{1, \ldots, m\}$, $a_i', a_i'' \in A_i$. The convex combination $\hat{a}$ is the m-tuple each of whose i-th elements is defined by $\hat{a}_i \equiv \alpha a_i' + (1-\alpha)a_i''$. Each of these $\hat{a}_i$ is a convex combination of a_i' and a_i'' and hence belongs to A_i because A_i is assumed to be convex. Therefore the original convex combination $\hat{a}$ is an m-tuple each of whose i-th elements belongs to A_i and therefore $\hat{a} \in A$. ☺

Now we need to show that the best-response correspondence ψ is nonempty valued, convex valued, and upper hemicontinuous. We have earlier seen that, for all players $i \in I$ and all deleted mixed-strategy profiles $\boldsymbol{\sigma}_{-i}$, the pure-strategy best-response correspondence $\mathsf{BR}_i(\boldsymbol{\sigma}_{-i})$ is nonempty.[18] Therefore there

15 Good references for fixed-point theorems are Border [1985], Debreu [1959], and Green and Heller [1981].

16 A subset of a Euclidean space is *compact* if and only if it is closed and bounded.

17 To say that the correspondence $\varphi\colon K \twoheadrightarrow K$ is nonempty valued means that $\forall x \in K$, $\varphi(x) \neq \varnothing$. To say that φ is convex valued is to say that $\forall x \in K$, $\varphi(x)$ is a convex set. We will define upper hemicontinuity soon!

18 See the handout "Strategic-Form Games."

exists a (possibly degenerate) best-response mixed strategy σ_i such that $\operatorname{supp}\sigma_i \subset \mathrm{BR}_i(\boldsymbol{\sigma}_{-i})$; hence, for all $i\in I$ and all $\boldsymbol{\sigma}\in\Sigma$, $\psi_i(\boldsymbol{\sigma})\neq\varnothing$ and therefore $\psi(\boldsymbol{\sigma})\neq\varnothing$.

Now we show that the best-response correspondence ψ is convex valued. We do this by first showing that, for each $i\in I$, ψ_i is convex valued, i.e. for all $\boldsymbol{\sigma}\in\Sigma$, $\psi_i(\boldsymbol{\sigma})$ is a convex set. Then $\psi(\boldsymbol{\sigma})$ is convex by the above lemma because it is the Cartesian product of the $\psi_i(\boldsymbol{\sigma})$. Consider two player-$i$ mixed strategies $\sigma_i', \sigma_i''\in\Sigma_i$ both of which are best responses to the deleted mixed-strategy profile $\boldsymbol{\sigma}_{-i}$ extracted from some mixed-strategy profile $\boldsymbol{\sigma}\in\Sigma$, i.e. $\sigma_i', \sigma_i''\in\psi_i(\boldsymbol{\sigma})$. We need to show that any convex combination of these two mixed strategies is also a best response to $\boldsymbol{\sigma}_{-i}$, i.e. for all $\alpha\in[0,1]$, $\hat{\sigma}_i\equiv[\alpha\sigma_i'+(1-\alpha)\sigma_i'']\in\psi_i(\boldsymbol{\sigma})$. Clearly this holds for $\alpha\in\{0,1\}$, so we focus on $\alpha\in(0,1)$. Because $\forall\sigma_i', \sigma_i''\in\psi_i(\boldsymbol{\sigma}_i)$, $\operatorname{supp}\sigma_i'\subset\mathrm{BR}_i(\boldsymbol{\sigma}_{-i})$ and $\operatorname{supp}\sigma_i''\subset\mathrm{BR}_i(\boldsymbol{\sigma}_{-i})$. For $\alpha\in(0,1)$,

$$\operatorname{supp}\hat{\sigma}_i=\operatorname{supp}\sigma_i'\cup\operatorname{supp}\sigma_i''\subset\mathrm{BR}_i(\boldsymbol{\sigma}_{-i}).$$ [19,20]

Therefore $\hat{\sigma}_i\in\psi_i(\boldsymbol{\sigma})$; therefore $\psi_i(\boldsymbol{\sigma})$ is a convex set; therefore $\psi(\boldsymbol{\sigma})$ is a convex set; and therefore ψ is convex valued.

Now we show that the best-response correspondence ψ is upper hemicontinuous.[21]

Definition Let $\varphi\colon K\twoheadrightarrow K$ be a correspondence where K is compact. Then φ is *upper hemicontinuous* if φ has a closed graph. I.e. φ is upper hemicontinuous if all convergent sequences in the graph of the correspondence converge to a point in the graph of the correspondence. I.e. if [for every sequence $\{(x^k,y^k)\}$ in $K\times K$ and point $(x^0,y^0)\in K\times K$ such that $(x^k,y^k)\to(x^0,y^0)$ and such that, for all k, $y^k\in\varphi(x^k)$], then it is the case that $y^0\in\varphi(x^0)$.

To show that ψ is upper hemicontinuous we assume to the contrary that there exists a convergent sequence of pairs of mixed-strategy profiles $(\boldsymbol{\sigma}^k,\hat{\boldsymbol{\sigma}}^k)\to(\boldsymbol{\sigma},\hat{\boldsymbol{\sigma}})$ such that, for all $k\in\mathbb{N}\equiv\{1,2,\ldots\}$, $\hat{\boldsymbol{\sigma}}^k\in\psi(\boldsymbol{\sigma}^k)$ but $\hat{\boldsymbol{\sigma}}\notin\psi(\boldsymbol{\sigma})$. Along the sequence, because $\hat{\boldsymbol{\sigma}}^k\in\psi(\boldsymbol{\sigma}^k)$, for each player $i\in I$,

$$\hat{\sigma}_i^k\in\psi_i(\boldsymbol{\sigma}^k). \tag{10}$$

At the limit, because $\hat{\boldsymbol{\sigma}}$ is not a best response to $\boldsymbol{\sigma}$, some player i must have a better strategy than $\hat{\sigma}_i$ against $\boldsymbol{\sigma}_{-i}$, i.e. because $\hat{\boldsymbol{\sigma}}\notin\psi(\boldsymbol{\sigma})$, $\exists i\in I$, $\exists\sigma_i'\in\Sigma_i$, such that

$$u_i(\sigma_i',\boldsymbol{\sigma}_{-i})>u_i(\hat{\sigma}_i,\boldsymbol{\sigma}_{-i}). \tag{11}$$

We now exploit the continuity of player i's utility function.[22] Because $\boldsymbol{\sigma}^k\to\boldsymbol{\sigma}$, it is true that $\boldsymbol{\sigma}_{-i}^k\to\boldsymbol{\sigma}_{-i}$,

19 Consider a pure strategy $s_i\in S_i$. Then $s_i\in\operatorname{supp}[\alpha\sigma_i'+(1-\alpha)\sigma_i'']$ if $\alpha\sigma_i'(s_i)+(1-\alpha)\sigma_i''(s_i)>0$. Because $\alpha\in(0,1)$, this occurs if and only if either $\sigma_i'(s_i)>0$ or $\sigma_i''(s_i)>0$.

20 For any sets A, B, and C, $(A\subset C$ and $B\subset C)$ implies $(A\cup B)\subset C$.

21 Hemicontinuity of correspondences has in the past also been called semicontinuity. Hemicontinuity is the more modern usage and intended to prevent any confusion with semicontinuity of functions. Debreu [1959], Border [1985], and Green and Heller [1981] are good references concerning the continuity of correspondences.

22 The function u_i is a continuous function of continuous functions.

and therefore we can take k sufficiently large to make $u_i(\sigma_i{}', \boldsymbol{\sigma}^k_{-i})$ arbitrarily close to left-hand side of (11), viz. $u_i(\sigma_i{}', \boldsymbol{\sigma}_{-i})$. Because $(\boldsymbol{\sigma}^k, \hat{\boldsymbol{\sigma}}^k) \to (\boldsymbol{\sigma}, \hat{\boldsymbol{\sigma}})$, we can take k sufficiently large to make $u_i(\hat{\sigma}^k_i, \boldsymbol{\sigma}^k_{-i})$ arbitrarily close to the right-hand side of (11), viz. $u_i(\hat{\sigma}_i, \boldsymbol{\sigma}_{-i})$. Therefore for all k sufficiently large we have

$$u_i(\sigma_i{}', \boldsymbol{\sigma}^k_{-i}) > u_i(\hat{\sigma}^k_i, \boldsymbol{\sigma}^k_{-i})\,. \tag{12}$$ [23]

But this is tantamount to saying that $\hat{\sigma}^k_i$ is not a best response to $\boldsymbol{\sigma}^k_{-i}$, and this contradicts (10). Therefore ψ must be upper hemicontinuous.

So we have verified that the space Σ of mixed-strategy profiles and the best-response correspondence ψ satisfy the hypotheses of Kakutani's fixed-point theorem. Therefore the best-response correspondence has a fixed point and therefore every game has a Nash equilibrium. ☺

23 The general argument here is the following: Let $x > y$ and let $\{x^k\}$ and $\{y^k\}$ be sequences such that $x^k \to x$ and $y^k \to y$. Then there exists a $\bar{k}$ such that, for all $k > \bar{k}$, $x^k > y^k$.

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