Finding Mixed-strategy Nash Equilibria in 2 × 2 Games

Introduction

We’ll now see explicitly how to find the set of (mixed-strategy) Nash equilibria for two-player games where each player has a strategy space containing two actions (i.e. a “2 × 2 matrix game”). After setting up the analytical framework and deriving some general results for such games, we will apply this technique to two particular games. The first game is a typical and straightforwardly solved example; the second is nongeneric in the sense that it has an infinite number of equilibria. For each game we will compute the graph of each player’s best-response correspondence and identify the set of Nash equilibria by finding the intersection of these two graphs.

The canonical game

We consider the two-player strategic-form game in Figure 1. We assign rows to player A and columns to player B. A’s strategy space is $S_A = \{U, D\}$ and B’s is $S_B = \{l, r\}$. Because each player has only two actions, each of her mixed strategies can be described by a single number ($p$ for A and $q$ for B)


1 Here nongeneric means that the phenomenon depends very sensitively on the exact payoffs. If the payoffs were perturbed the slightest bit, then the phenomenon would disappear.
belonging to the unit interval \([0, 1]\). A mixed-strategy profile for this game, then, is an ordered pair \((p, q)\) \(\in [0, 1] \times [0, 1]\). We denote the players’ payoffs resulting from pure-strategy profiles by subscripted \(a\)’s and \(b\)’s, respectively. E.g. the payoff for \(A\) when \(A\) plays \(D\) and \(B\) plays \(r\) is \(a_{Dr}\).

\[
\begin{array}{c|cc}
 & l: q & r: 1-q \\
\hline
U: p & a_{Ul}, b_{Ul} & a_{Ur}, b_{Ur} \\
A & 1-p & a_{Dr}, b_{Dr} \\
D: & & \\
\end{array}
\]

Figure 1: The canonical two-player, two-action-per-player strategic-form game.

The pure-strategy equilibria, if any, of such a game are easily found by inspection of the payoffs in each cell, each cell corresponding to a pure-strategy profile. A particular pure-strategy profile is a Nash equilibrium if and only if 1 that cell’s payoff to the row player (viz. \(A\)) is a (weak) maximum over all of \(A\)’s payoffs in that column (otherwise the row player could profitably deviate by picking a different row given \(B\)’s choice of column) and 2 that cell’s payoff to the column player (viz. \(B\)) is a (weak) maximum over all of \(B\)’s payoffs in that row. For example, the pure-strategy profile \((U, r)\) would be a Nash equilibrium if and only if the payoffs were such that \(a_{Ur} \geq a_{Dr}\) and \(b_{Ur} \geq b_{Dr}\).

**Best-response correspondences**

Finding the pure-strategy equilibria was immediate. Finding the mixed-strategy equilibria takes a little more work, however. To do this we need first to find each player’s best-response correspondence. We will show in detail how to compute player \(A\)’s correspondence. Player \(B\)’s is found in exactly the same way.

Player \(A\)’s best-response correspondence specifies, for each mixed strategy \(q\) played by \(B\), the set of mixed strategies \(p\) which are best responses for \(A\). I.e. it is a correspondence \(p^*\) which associates with every \(q \in [0, 1]\) a set \(p^*(q) \subset [0, 1]\) such that every element of \(p^*(q)\) is a best response by \(A\) to \(B\)’s choice \(q\). The graph of \(p^*\) is the set of points

\[
\{(p, q): q \in [0, 1], p \in p^*(q)\}.
\]

(1)

**\(A\)’s payoff as a function of the mixed-strategy profile**

To find \(A\)’s best-response correspondence we first compute her expected payoff for an arbitrary mixed-strategy profile \((p, q)\) by weighting each of \(A\)’s pure-strategy profile payoffs by the probability of that profile’s occurrence as determined by the mixed-strategy profile \((p, q)\):\(^1\)

\[
u_A(p, q) = pq a_{Ul} + p(1-q)a_{Ur} + (1-p)q a_{Dr} + (1-p)(1-q)a_{Dr}.
\]

(2)

\(A\)’s utility maximization problem is

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\(^1\) The semicolon in “\(u(p; q)\)” is used to denote that, while \(p\) is a choice variable for \(A\), \(q\) is a parameter outside of \(A\)’s control.
\[
\max_{p \in [0,1]} u_A(p; q)
\]  
(3)

Because \( p \) is \( A \)'s choice variable, it will be convenient to rewrite equation (2) as an affine function of \( p \):

\[
u_A(p; q) = p[(a_{U1} - a_{Ur} - a_{Dr} + a_{Dl})q + (a_{Dr} - a_{Ur})] + [(a_{Dl} - a_{Dr})q + a_{Dr}],
\]  
(4a)

\[
\delta(q)p + [(a_{Dl} - a_{Dr})q + a_{Dr}].
\]  
(4b)

Of interest here is the sign of the coefficient of \( p \),

\[
\delta(q) \equiv (a_{U1} - a_{Ur} - a_{Dl} + a_{Dr})q + (a_{Ur} - a_{Dl}),
\]  
(5)

which is itself an affine function of \( q \).

\( A \)'s best-response correspondence

For a given \( q \), the function \( u_A(p; q) \) will be maximized with respect to \( p \) either 1 at the unit interval’s right endpoint (viz. \( p = 1 \)) if \( \delta(q) \) is positive, 2 at the interval’s left endpoint (viz. \( p = 0 \)) if \( \delta(q) \) is negative, or 3 for every \( p \in [0,1] \) if \( \delta(q) \) is zero, because \( u_A(p; q) \) is then constant with respect to \( p \).

Now we consider the behavior of \( A \)'s best response as a function of \( q \). There are three major cases to consider.

**Case 1: complete indifference**

\( A \)'s payoffs could be such that \( \delta(q) = 0 \) for all \( q \).\(^2\) In this case \( A \)'s best-response correspondence would be independent of \( q \) and would simply be the unit interval itself: \( \forall q \in [0,1], \ p^*(q) = [0,1] \). In other words \( A \) would be willing to play any mixed strategy regardless of \( B \)'s choice of strategy. The graph of \( p^* \) in this case is the entire unit square \( [0,1] \times [0,1] \). (See Figure 2.)

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1. I would be happy to say linear here instead of affine. The strict definition of linear seems to be made consistently in linear algebra, but the less restrictive definition seems to be tolerated in other contexts.
2. This would require that in each column \( A \) receives the same payoff in each of the two rows.
Case 2: A has a dominant pure strategy

If this is not the case, i.e. if $\delta(q)$ is not identically zero, then—because $\delta(q)$ is affine—there will be exactly one value $q^\dagger$ at which $\delta(q^\dagger)=0$. For all $q$ to one side of $q^\dagger$, $\delta(q)$ will be positive; for all $q$ on the other side of $q^\dagger$, $\delta(q)$ will be negative. However, this $q^\dagger$ need not be an element of $[0, 1]$. If $q^\dagger \notin [0, 1]$, then all $q \in [0, 1]$ will lie on a common side of $q^\dagger$ and therefore $\delta(q)$ will have a single sign throughout the interval $[0, 1]$. Therefore A will have the same best response for every $q$ (viz. $p = 1$ if $\delta(q) > 0$ on $[0, 1]$; $p = 0$, if $\delta(q) < 0$ on $[0, 1]$); i.e. A has a strongly dominant pure strategy. (See Figure 3.)

Case 3: A plays strategically

Now we consider the case where A plays strategically: her optimal strategy depends upon her opponent’s strategy. If $q^\dagger \in (0, 1)$, then the unit interval is divided into two subintervals—those points to the right of $q^\dagger$ and those points to the left of $q^\dagger$—in each of which A plays a different pure strategy. At exactly $q^\dagger$, A can mix with any $p \in [0, 1]$. (See Figure 4.)
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If $q^\dagger = 0$ or $q^\dagger = 1$, then $A$ can mix at $q^\dagger$ and will play a common pure strategy for every other $q \in [0, 1]$. In this case the pure strategy which $A$ plays against $q \in (0, 1)$ is a weakly dominant strategy. (It is strictly best against $q \in [0, 1] - \{q^\dagger\}$ and as good as anything else against $q^\dagger$.)

**Summary of the best-response correspondence**

We see that—in a $2 \times 2$ matrix game—each player will either 1 be free to mix regardless of her opponent’s strategy (a case which ‘almost never’ occurs), 2 play a dominant pure strategy, or 3 be free to mix in response to exactly one of her opponent’s possible choices of strategy but for every other choice will play a pure strategy.

**$B$’s best-response correspondence**

Player $B$’s best-response correspondence specifies, for each mixed strategy $p$ played by $A$, the set of mixed strategies $q$ which are best responses for $B$. I.e. it is a correspondence $q^*$ which associates with
every \( p \in [0, 1] \) a set \( q^*(p) \subset [0, 1] \) such that every element of \( q^*(p) \) is a best response by \( B \) to \( A \)’s choice \( p \). The graph of \( q^* \) is the set of points

\[
\{(p, q): p \in [0, 1], \quad q \in q^*(p)\}.
\] (6)

This correspondence is found using the same method of analysis we used for \( A \)’s. You will easily show that

\[
u_B(q; p) = \gamma(p)q + [(b_{Ur} - b_{Dr})p + b_{Dr}],
\] (7)

where

\[
\gamma(p) = (b_{U1} - b_{Ur} - b_{Dr} + b_{Dr})p + (b_{Dr} - b_{Dr}).
\] (8)

The Nash equilibria are the points in the intersection of the graphs of \( A \)’s and \( B \)’s best-response correspondences

We know that a mixed-strategy profile \( (p, q) \) is a Nash equilibrium if and only if \( 1 \) \( p \) is a best response by \( A \) to \( B \)’s choice \( q \) and \( 2 \) \( q \) is a best response by \( B \) to \( A \)’s choice \( p \). We see from (1) that the first requirement is equivalent to \( (p, q) \) belonging to the graph of \( p^* \), and from (6) we see that the second requirement is equivalent to \( (p, q) \) belonging to the graph of \( q^* \). Therefore we see that \( (p, q) \) is a Nash equilibrium if and only if it belongs to the intersection of the graphs of the best-response correspondences \( p^* \) and \( q^* \). We can write the set of Nash equilibria, then, as

\[
\{(p, q) \in [0, 1] \times [0, 1]: p \in p^*(q), \quad q \in q^*(p)\}.
\] (9)

A typical example

Consider the \( 2 \times 2 \) game in Figure 5. First we immediately observe that there are two pure-strategy Nash equilibria: \( (U, r) \) and \( (D, l) \).

\[
\begin{array}{c|cc}
 & l & r \\
\hline
U & \begin{cases} 1, -1 \\ 3, 0 \end{cases} & \begin{cases} 1-q \\ 1-q \end{cases} \\
D & \begin{cases} 4, 2 \\ 0, -1 \end{cases} & \begin{cases} 1-q \end{cases}
\end{array}
\]

Figure 5: A typical \( 2 \times 2 \) game.

\( A \)’s best-response correspondence

Now we find \( A \)’s best-response correspondence. From (5) we see that

\[
\delta(q) = -6q + 3,
\] (10)

which vanishes at \( q^+ = \frac{1}{2} \). Because \( \delta(q) \) is decreasing in \( q \) we see that \( A \) will choose the pure strategy \( p = 1 \) against \( q \)’s on the interval \([0, \frac{1}{2}]\) and the pure strategy \( p = 0 \) against \( q \)’s on the interval \((\frac{1}{2}, 1]\). Against \( q = q^+ = \frac{1}{2} \), \( A \) is free to choose any mixing probability. Player \( A \)’s best-response correspondence \( p^* \) is plotted in Figure 6.
We can also derive A’s best-response correspondence graphically by plotting her payoff to her different pure strategies as a function of B’s mixed-strategy choice \( q \). Using (4b) and (10) we have

\[
    u_A(p; q) = (-6q + 3)p + 4q. \tag{11}
\]

Evaluating this payoff function at A’s pure-strategy choices \( p = 1 \) and \( p = 0 \), respectively, we have

\[
    u_A(U; q) = 3 - 2q, \tag{12}
\]

\[
    u_A(D; q) = 4q. \tag{13}
\]

Both of these functions are plotted for \( q \in [0, 1] \) in Figure 7. These two lines intersect when \( q = q^* = \frac{1}{2} \); i.e. they intersect at the mixed strategy for B at which we earlier determined A would be willing to mix. To the left of this point, A’s payoff to U is higher than her payoff to D; the reverse is true on the other side of \( q^* \). Therefore A’s best response is to play U against \( q \in [0, \frac{1}{2}] \) and D against \( q \in (\frac{1}{2}, 1] \). At the intersection point \( q^* \), A is indifferent to playing U or D, so she is free to mix between them. This is exactly the best-response correspondence we derived analytically above.

**B’s best-response correspondence**

We similarly find B’s best-response correspondence. From (8) we find that

\[
    
\]

which decreases in \( p \) and vanishes at \( p^\dagger = \frac{3}{4} \). Player B, then, chooses the pure strategy \( q = 1 \) against \( p^\dagger \)'s on the interval \( [0, \frac{3}{4}] \) and the pure strategy \( q = 0 \) against \( p^\dagger \)'s on the interval \( (\frac{3}{4}, 1] \). Against \( p = p^\dagger = \frac{3}{4} \),
$B$ is free to choose any mixing probability. Player $B$’s best-response correspondence $q^*$ is plotted in Figure 8.

Figure 7: $A$’s pure-strategy payoffs as a function of $B$’s mixed strategy $q$.

Figure 8: $B$’s best-response correspondence for the game of Figure 5.
Using (7) and (14) we compute $B$’s payoff to the mixed-strategy profile $(p, q)$ to be
\[ u_B(q; p) = (-4p + 3)q + (p - 1). \] (15)
Evaluating this at $B$’s pure-strategy choices we get
\[ u_B(l; p) = 2 - 3p, \] (16)
\[ u_B(r; p) = p - 1. \] (17)
These functions are plotted for $p \in [0, 1]$ in Figure 9. We again note that their intersection—in this case at $p^* = 3/4$—occurs at the opponent’s mixed strategy which allows mixing by the player choosing a best response. The same reasoning we used above to graphically derive $A$’s best-response correspondence works here, and we arrive at the same behavior rules we found analytically.

![Figure 9: $B$’s pure-strategy payoffs as a function of $A$’s mixed strategy $p$.](image)

The Nash set

Both $A$’s and $B$’s best-response correspondences are plotted together in Figure 10. We see that the intersection of the graphs of the two best-response correspondences contains exactly three points, each corresponding to a mixed-strategy profile $(p, q)$: $(0, 1)$, $(3/4, 1/2)$, and $(1, 0)$. The first and last of these correspond to the two pure-strategy Nash equilibria we identified earlier. Note that the additional equilibrium we found is the strategy profile $(p^*, q^*)$. This strategy profile will in general be the only
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non-pure equilibrium strategy profile when $p^\dagger$ and $q^\dagger$ both lie in the interior of the unit interval.¹ Note as well that there is an odd number of Nash equilibria for this game, as is “almost always” the case. The payoff vectors for these equilibria are $(4, 2), (2, -\frac{1}{4})$, and $(3, 0)$, respectively. [The mixed-strategy profile payoffs are computed using (11) and (15).] Note that the equilibrium payoffs are completely Pareto ranked.

A nongeneric example

We now consider the two-player normal form game in Figure 11. We immediately determine that the unique pure-strategy equilibrium is $(D, r)$.

![Figure 10: The players’ best-response correspondences, the Nash set, and equilibrium payoffs.](image)

To find A’s best-response correspondence we use (5) to compute that

$$\delta(q) = -3q,$$  \hspace{1cm}  (18)

¹ It is not possible when $p^\dagger, q^\dagger \in (0, 1)$ that in equilibrium only one player mixes. Assume, say, that A mixes. We know that A will mix only when $q = q^\dagger \in (0, 1)$. Therefore B is mixing as well.
which decreases in \( q \) and vanishes at \( q^\dagger = 0 \). Therefore \( A \) plays the pure strategy \( p = 0 \) against any \( q \in (0, 1] \) and she is willing to mix against \( q = q^\dagger = 0 \). Therefore \( D \) is a weakly dominant strategy for \( A \). Her best-response correspondence \( p^*(q) \) is plotted in Figure 12.

To find \( B \)'s best-response correspondence we use (8) and find that
\[
\gamma(p) = \frac{3}{2} p - 1.
\]  
(19)

This function increases in \( p \) and vanishes at \( p^\dagger = 2/3 \). Therefore \( B \) will play the pure strategy \( q = 0 \) against any \( p \in [0, 2/3) \), will play the pure strategy \( q = 1 \) against any \( p \in (2/3, 1] \), and will be free to mix for \( p = p^\dagger = 2/3 \). \( B \)'s best-response correspondence \( q^*(p) \) is also plotted in Figure 12.

Inspection of Figure 12 shows that the intersection of the graphs of \( A \)'s and \( B \)'s best-response correspondences is a line segment along which \( B \) plays \( q = 0 \) and \( A \) mixes with any probability \( p \) on \([0, 2/3]\). We note that the unique pure-strategy Nash equilibrium we identified earlier is the left endpoint of this set. This example is nongeneric in that we have an infinity (in fact, a continuum) of equilibria: a situation which generically never happens.

Before leaving this example we should also take note of the equilibrium payoffs. At the pure-strategy equilibrium each player gets 3. (See Figure 12.) Player \( A \) is indifferent to mixing between \( U \) and \( D \), given that \( B \) is playing \( r \). However, this mixing hurts \( B \). At the right-hand endpoint of the Nash set, \( A \) still receives 3 but \( B \)'s payoff, which has been decreasing linearly with \( A \)'s mixing probability \( p \), has declined to \( 5/3 \). Note that, at the alternative strategy profile in which \( B \) plays \( l \) and \( A \) mixes with \( p = 2/3 \), \( B \) would get the same payoff as playing \( r \), but \( A \) would get only 2, rather than 3. If \( A \) mixed any more
strongly toward $U$ than $p = \frac{2}{3}$, $B$ would defect to the alternative strategy $l$, giving $A$ less than in this equilibrium. This is what determines the location of the right-hand endpoint of the Nash set. In this game the set of equilibria are Pareto ranked. Obviously, player $B$ would prefer to coordinate on the pure-strategy equilibrium, and there is no reason $A$ should disagree.